Q1: Find and sketch the domain of the function.

(a) $f(x,y)=\ln(x+y+1)$

(b) $f(x,y)=\sqrt{x^2+y^2-4}$

(c) $f(x,y)=\sqrt{(x-4)(y-9)}$

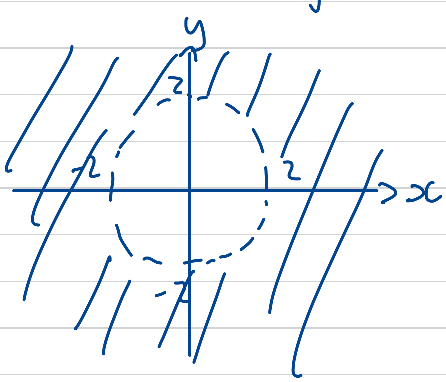
(d) $f(x,y)=\dfrac{\sqrt{y-x^2}}{(x^2-1)\ln(x+y-1)}$

a) $f(x,y) = \ln(x+y+1)$

$x+y+1 > 0 \Rightarrow x+y > -1$

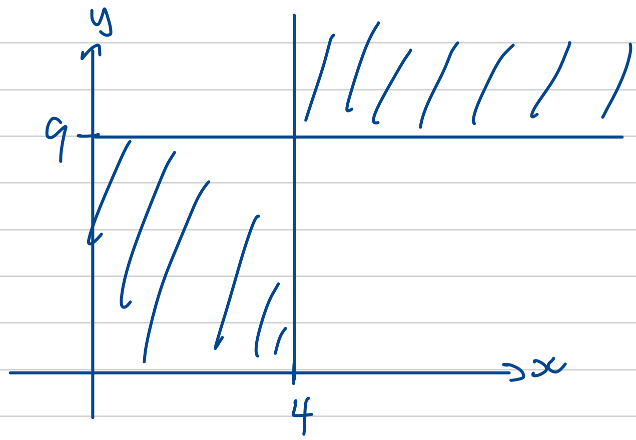
b) $f(x,y) = \sqrt{x^2+y^2-4}$

$x^2+y^2-4 \geq 0 \Rightarrow x^2+y^2 \geq 4$



c) $f(x,y) = \sqrt{(x-4)(y-9)}$

$(x-4)(y-9) \geq 0 \Rightarrow \begin{cases} x \geq 4 \quad y \geq 9 \quad \text{or} \\ x \leq 4 \quad y \leq 9 \end{cases}$

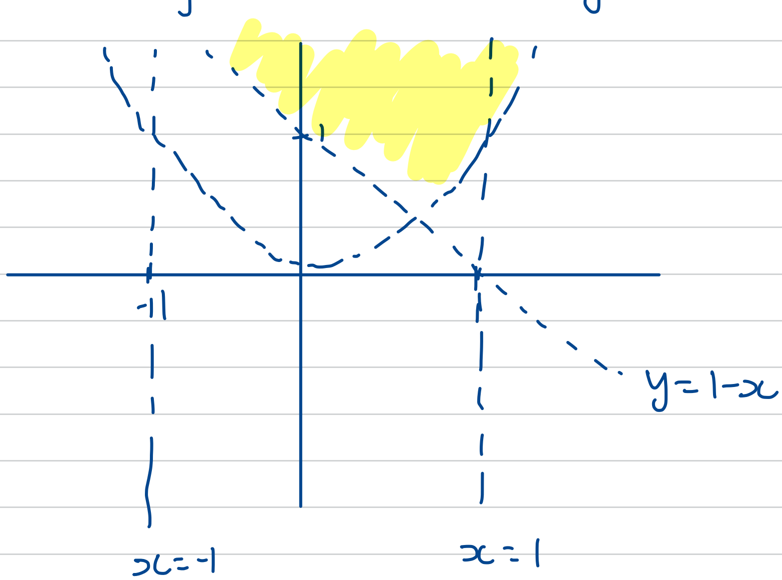


d) $f(x,y) = \dfrac{\sqrt{y-x^2}}{(x^2-1)\ln(x+y-1)}$

Denom: $x^2-1 \neq 0 \implies x \neq \pm 1$

$x+y-1 > 0 \implies x+y > 1$

Num: $y - x^2 \geq 0 \implies y \geq x^2$



Q2: Sketch the level curves of the function.

(a) $f(x,y)=\sqrt{4x^2+y^2}$

(b) $f(x,y)=|x+1|-|y+1|$

(c) $f(x,y)=xy$

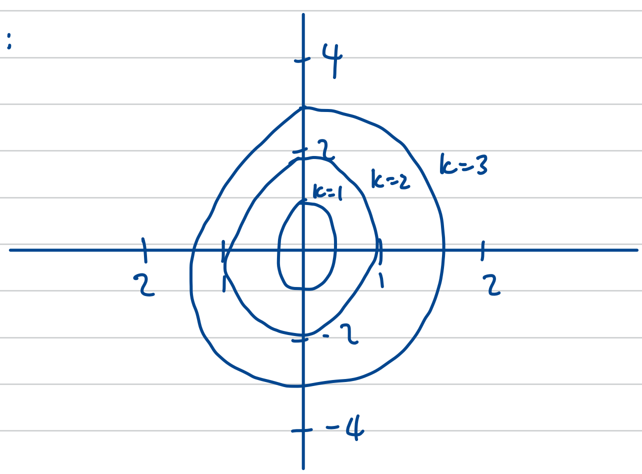
(d) $f(x,y)=e^x+y$

a) $f(x,y)=\sqrt{4x^2+y^2}=k$ for $k\geq 0$

$\Rightarrow 4x^2+y^2=k^2$

$\Rightarrow \dfrac{x^2}{(k/2)^2}+\dfrac{y^2}{k^2}=1$

Ellipse:



b) $f(x,y)=|x+1|-|y+1|=k$

Case 1: $x+1\geq 0$ $y+1\geq 0$

$|x+1|-|y+1|=x+1-(y+1)=x-y$

$\Rightarrow x-y=k$

Case 2: $x+1<0$ $y+1\geq 0$

$|x+1|-|y+1|=-(x+1)-(y+1)$

$\Rightarrow -x-y-2=k$

Case 3: $x+1 \geq 0 \quad y+1<0$

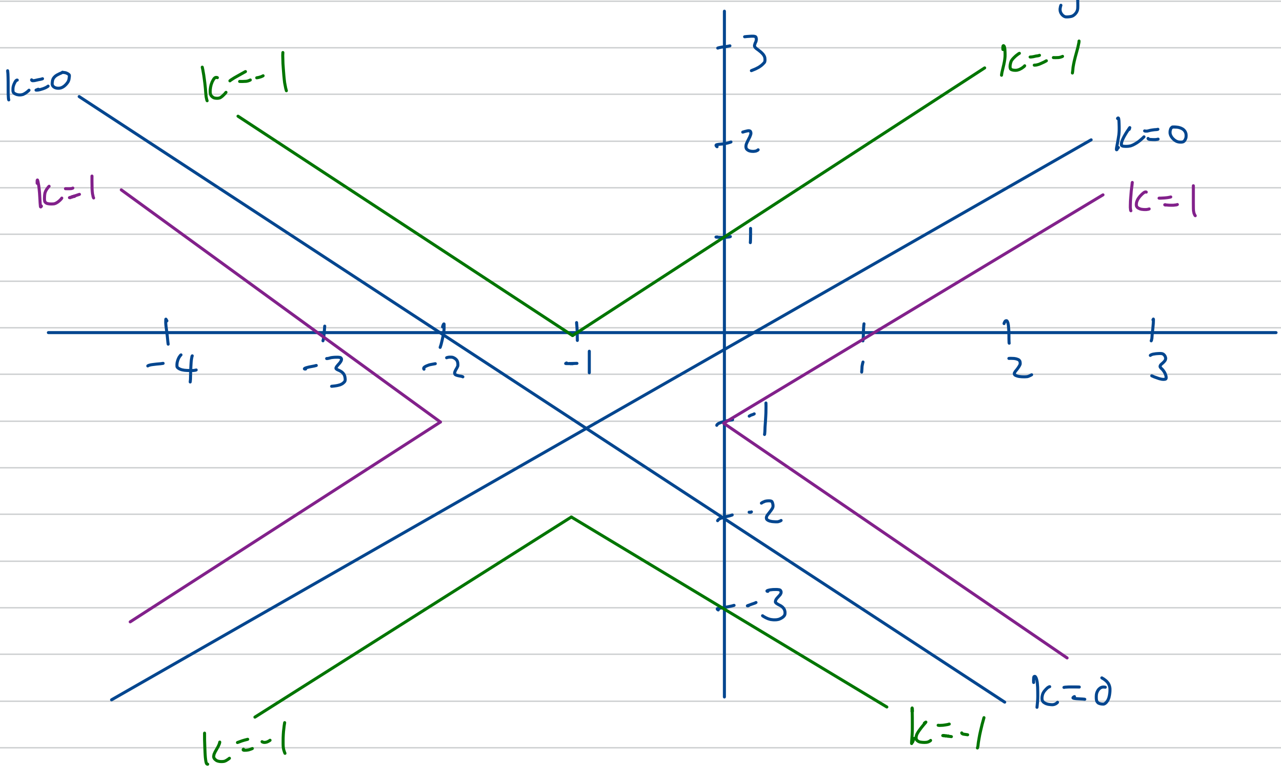
$$|x+1|-|y+1| = x+1-(-(y+1)) = x+y+2$$

$$\Rightarrow x+y+2=k$$

Case 4: $x+1<0 \quad y+1<0$

$$|x+1|-|y+1| = -(x+1)-(-(y+1)) = y-x$$

$$\Rightarrow y-x=k$$



Q4: Find all second partial derivatives.

(a) $f(x,y)=4x^3-xy^2$ (b) $v=r\cos(s+2t)$

(c) $f(x,y,z)=x^k y^m z^n$ (d) $z=xe^{-2y}$

a) $f_x = 12x^2-y^2$

$f_y = -2yx$

$\Rightarrow$

$f_{xx} = 24x$

$f_{xy} = -2y$

$f_{yy} = -2x$

$f_{yx} = -2y$

b) $V_s = -r\sin(s+2t)$

$V_t = -2r\sin(s+2t)$

$V_r = \cos(s+2t)$

$V_{ss} = -r\cos(s+2t)$

$V_{rr} = 0$

$V_{tt} = -4r\cos(s+2t)$

Clairaut's Thm

$V_{st} = -2r\cos(s+2t) = V_{ts}$

$V_{rs} = -\sin(s+2t) = V_{sr}$

$V_{rt} = -2\sin(s+2t) = V_{tr}$

Q5: if $z = xy + xe^{1/x}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z + xy - e^{1/x}$.

$$\frac{\partial z}{\partial x} = y + \left(e^{1/x} + xe^{1/x}\left(-\frac{1}{x^2}\right)\right)$$

$$= y + e^{1/x}\left(1 - \frac{1}{x}\right)$$

$$\frac{\partial z}{\partial y} = x$$

LHS: $x\left(y + e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right)\right) + y(x)$

$$= xy + xe^{1/x}\left(1 - \frac{1}{x}\right) + xy$$

$$= \underbrace{xy + xe^{1/x}} - e^{1/x} + xy$$

$$= z + xy - e^{1/x}$$

Q6: Find all value(s) of the constant α such that $u(x,t) = t^{-1/2}e^{-x^2/t}$ satisfies the equation $\dfrac{\partial u}{\partial t} = \alpha\dfrac{\partial^2 u}{\partial x^2}$.

$$\cdot\ \frac{\partial u}{\partial t} = -\frac{1}{2}t^{-\frac{3}{2}}e^{-x^2/t} + t^{-\frac{1}{2}}e^{-x^2/t}\left(\frac{x^2}{t^2}\right)$$

$$= e^{-x^2/t}\left(-\frac{1}{2}t^{-\frac{3}{2}} + \frac{x^2}{t^{5/2}}\right)$$

- $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x}\left(-\frac{2x}{t^{3/2}} e^{-x^2/t}\right)$$

$$= \left(\frac{-2t + 4x^2/t}{t^{5/2}}\right)\left(e^{-x^2/t}\right) = \left(-\frac{2}{t^{3/2}} + \frac{4x^2}{t^{5/2}}\right)$$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \cancel{e^{-x^2/t}}\left(-\frac{1}{2t^{3/2}} + \frac{x^2}{t^{5/2}}\right) = \left(-\frac{2\alpha}{t^{3/2}} + \frac{4x^2\alpha}{x^{5/2}}\right)\cancel{e^{-x^2/t}}$$

$\Rightarrow$ Term by term matching $\quad -\frac{1}{2} = -2\alpha \quad , \quad 1 = 4\alpha$

$\Rightarrow \alpha = 1/4$

Q7: Find equations of the tangent plane to the given surface at the specified point.

(a) $z = 3x^2 - y^2 + 2x \quad , \quad (1,-2)$

$f_x = 6x + 2 \qquad f_y = -2y \qquad f(1,-2) = 1$

$f_x(1,-2) = 8 \qquad f_y(1,-2) = 4 \qquad \uparrow z_0$

Eqn: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$$z = 8(x-1) + 4(y+2) + 1$$