Q1: (a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

(b) Eliminate the parameter to find a Cartesian equation of the curve, if possible.

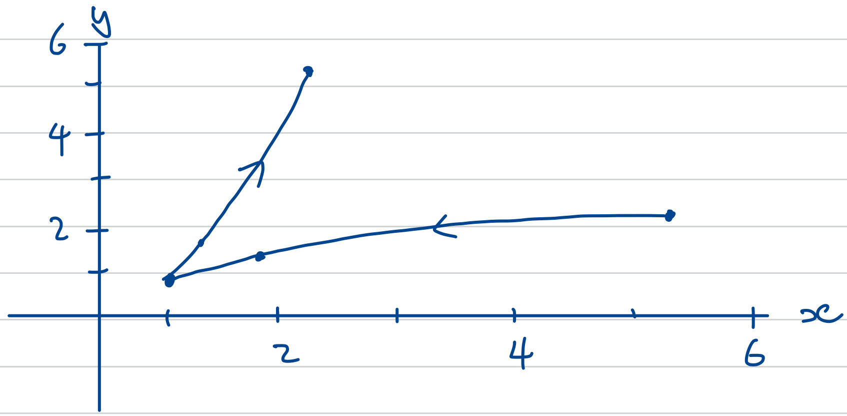
(i) $x = e^{-t} + t \quad , \quad y = e^{t} - t \quad , \quad -2 \le t \le 2$

(ii) $x = \cos(\theta) \quad , \quad y = \sec(\theta) \quad , \quad 0 \le \theta \le \pi/2$

(iii) $x = \dfrac{t}{t-1} \quad , \quad y = \dfrac{t-2}{t+1} \quad , \quad -1 < t < 1$

(i)

t	$x = e^{-t} + t$	$y = e^{t} - t$
-2	$e^{-2} - 2 = 5.39$	$e^{-2} + 2 = 2.135$
-1	1.72	1.368
0	1	1
1	1.368	1.72
2	2.135	5.39



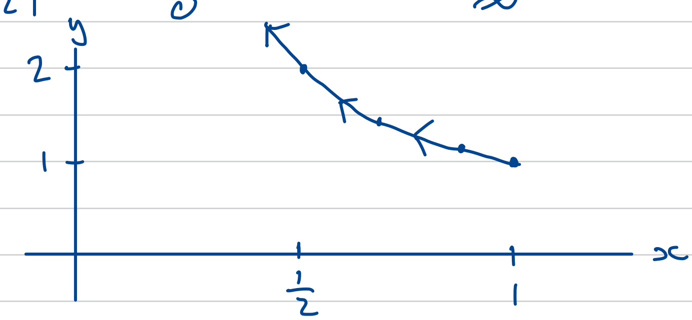
No closed Cartesian form

(ii)

θ	$x = \cos(\theta)$	$y = \sec(\theta) = 1/\cos(\theta)$
0	1	1
$\pi/6$	0.866	1.155
$\pi/4$	0.707	1.414
$\pi/3$	1/2	2
$\pi/2$	0	∞

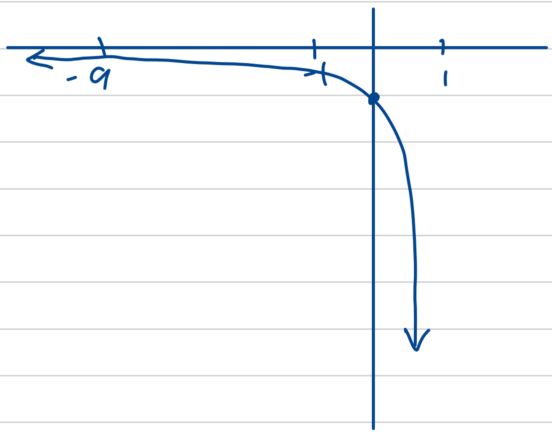
$$y = \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{x}$$

for $x \in (0, 1]$



(iii)

t	$x = t/t-1$	$y = t-2/t+1$
-0.9	0.474	-29
-0.5	0.333	-5
0	0	-2
0.5	-1	-1
0.9	-9	-0.579



$x = \dfrac{t}{t-1} \Rightarrow x(t-1) = t$

$\Rightarrow xt - x = t$

$\Rightarrow t(x-1) = x$

$\Rightarrow t = \dfrac{x}{x-1}$

$y = \dfrac{t-2}{t+1} = \dfrac{\frac{x}{x-1} - 2}{\frac{x}{x-1} + 1}$

$= \dfrac{x - 2(x-1)}{x + x - 1} = \dfrac{-x+2}{2x-1}$

where $x \neq 1$

Q2: Find equation of the tangent line to the curve at the given point.

(a) $x = t^2 - 1 \;,\; y = t^2 + t + 1 \;,\; (0,3)$

(b) $x = 1 + \sqrt{t} \;,\; y = e^{t^2} \;,\; (2,e)$

a) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{2t}$

Find t such that $x=0, y=3$

$x = t^2 - 1 = 0 \Rightarrow t = \pm 1$

- $t = -1$: $y = (-1)^2 + (-1) + 1 = 1 - 1 + 1 \neq 3$
- $t = 1$: $y = 1^2 + 1 + 1 = 3$

so $t = 1$

Now, $\left.\dfrac{dy}{dx}\right|_{t=1} = \dfrac{3}{2}$

Point-Slope: $y - 3 = \frac{3}{2}(x - 0)$

b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2te^{t^2}}{1/2\sqrt{t}} = 4t\sqrt{t}e^{t^2}$

Find t such that $x=2$, $y=e$

$x = 1+\sqrt{t} = 2 \Rightarrow t=1$
$y = e^{1^2} = e$

Now, $\left.\frac{dy}{dx}\right|_{t=1} = 4e$

Point-Slope: $y - e = 4e(x-2)$

Q3: Find $\frac{d^2y}{dx^2}$ at the given point.

(a) $x=1$, $y=\sqrt{t}$, $t=\frac{1}{4}$

(b) $x=\frac{1}{t}$, $y=-2+\ln(t)$, $t=1$

a) $\frac{dy}{dt} = \frac{1}{2}t^{-1/2}$ $\quad \frac{dx}{dt} = 0$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2\sqrt{t}}}{0} \Rightarrow$ undefined

$\therefore$ Vertical tangent $\Rightarrow d^2y/dx^2$ @ $t=1/4$ undefined

b) $\frac{dy}{dt} = \frac{1}{t}$ $\quad \frac{dx}{dt} = -\frac{1}{t^2}$ $\quad \therefore \frac{dy}{dx} = \frac{1/t}{-1/t^2} = -t$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(-t) = -1$, then

$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \left.\frac{\frac{d}{dt}(dy/dx)}{dx/dt}\right|_{t=1} = \left.\frac{-1}{-1/t^2}\right|_{t=1} = \left. t^2\right|_{t=1} = 1$

$\therefore d^2y/dx^2 = 1$ @ $t=1$

Q4: Find the arc length of the curve.

(a) $x = 3t^2$, $y = 2t^2$, $0 \le t \le 2$

(b) $x = 3\cos(t) - \cos(3t)$, $y = 3\sin(t) - \sin(3t)$, $0 \le t \le \pi$

(c) $x = \dfrac{(2t+3)^{3/2}}{3}$, $y = t + \dfrac{t^2}{2}$, $0 \le t \le 3$

a) $\dfrac{dx}{dt} = 6t \qquad \dfrac{dy}{dt} = 4t$

$$L = \int_0^2 \sqrt{(6t)^2 + (4t)^2}\, dt = \int_0^2 \sqrt{52t^2}\, dt$$

$$= \int_0^2 \sqrt{52}\, t\, dt = \sqrt{52}\, \frac{t^2}{2}\Big|_0^2 = \frac{4\sqrt{52}}{2} = 2\sqrt{52}$$

$$= 4\sqrt{13}$$

b) $\dfrac{dx}{dt} = -3\sin t + 3\sin(3t) \qquad \dfrac{dy}{dt} = 3\cos t - 3\cos(3t)$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-3\sin t + 3\sin(3t))^2 + (3\cos t - 3\cos(3t))^2$$

$$= 9\left([(\sin 3t - \sin t)]^2 + [\cos 3t - \cos t]^2\right)$$

Note: $[\sin A - \sin B]^2 + [\cos A - \cos B]^2 = 2[1 - \cos(A-B)]$
and $(X-Y)^2 = (Y-X)^2$

$$= 9 \cdot 2[1 - \cos(3t - t)]$$

$$= 18(1 - \cos 2t)$$

$$L = \int_0^\pi \sqrt{18(1-\cos 2t)}\, dt = \int_0^\pi \sqrt{18 \cdot 2\sin^2 t}\, dt = \int_0^\pi \sqrt{36\sin^2 t}\, dt$$

$$= \int_0^\pi 6|\sin t|\, dt = \int_0^\pi 6\sin t\, dt = 6 \cdot (-\cos t)\Big|_0^\pi$$

$$= 6(-\cos\pi + \cos 0) = 6(1 - \cos\pi)$$

c) $\frac{dx}{dt} = \frac{1}{3} \cdot \frac{3}{2} (2t+3)^{1/2} \cdot 2 = (2t+3)^{1/2}$

$\frac{dy}{dt} = 1+t$

$L = \int_0^3 \sqrt{(2t+3)+(1+t)^2}\, dt = \int_0^3 \sqrt{t^2+4t+4}\; dt$

$= \int_0^3 \sqrt{(t+2)^2}\; dt = \int_0^3 |t+2|\, dt = \int_0^3 t+2\, dt$

$= \left(\frac{t^2}{2}+2t\right)\Big|_0^3 = \frac{9}{2}+6 = \frac{21}{2}$